Electric routing and concurrent flow cutting

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ISAAC'09

Outline

- Oblivious routing, history, results
- Geometry and routing
- Electric flow and electric routing
- Congestion and L_1 spectral inequalities
- Remarks and other results

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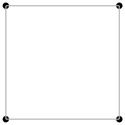
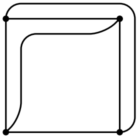
Oblivious routing problem

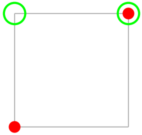
1 Graph instance



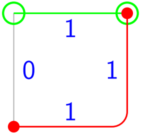
2 Oblivious routes



3 Adversarial demands

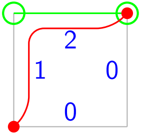


4 Optimal routing



$\|t_{\text{opt}}\|_\infty = 1$

5 Oblivious routing



$\|t_{\text{obl}}\|_\infty = 2$

Ratio $\eta = \max_{G,D} \dfrac{\|t_{\text{obl}}\|_\infty}{\|t_{\text{opt}}\|_\infty}$

Problem history

Input family	Type	Ratio $\frac{\|t_{\mathrm{obl}}\|_p}{\|t_{\mathrm{opt}}\|_p}$	Time	
hypercube	ℓ_2	$O(\log n)$	n/a	Valiant'81
any	ℓ_∞	$O(\log^3 n)$	$\exp(n)$	Räcke'02
any	ℓ_∞	$O(\log^2 n \cdot \log\log n)$	$\mathrm{poly}(n)$	Harrelson'03
any	$\ell_{1\leqslant p\leqslant\infty}$	$O(\log n)$	$\mathrm{poly}(n)$	Räcke'08'10
expanders	ℓ_∞	$O(\log n)$	$\tilde{O}(n)$	this work
"	$\ell_{1\leqslant p\leqslant\infty}$	"	n/a	Lawler'09

✤ Ratio lower-bound $\Omega(\log n/\log\log n)$ for expanders Hajiaghayi'06

New Algorithmics

- Computation
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  - Vertices = processors, edges = communication links
  - $O(\log n)$ rounds of communication (for expanders)
  - Asynchronous version as well
- Routing scheme representation
  - Per-vertex routing tables of size $\deg(v) \cdot n$
- Querying the routing scheme
  - At vertex v, given source s and sink t,
  - Compute the next hop in $O(1)$ time using local table

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  - New rounding techniques
  - Also see Lawler'09

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- Fault-tolerance = statements about distribution of edge-flow in electric current

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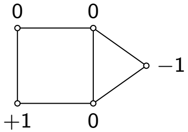
Demand and flow

- A demand is a vector like $d = \mathbb{1}_s - \mathbb{1}_t$
  - Formally, any $d \in \mathbb{R}^V$ with $\sum_v d_v = 0$



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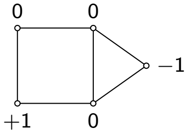

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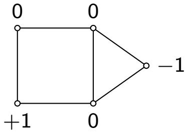
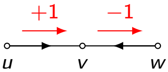


- Fix any orientation $u \to v$ on G's edges
- A flow is a vector $f \in \mathbb{R}^E$
  - Think $f_{(u,v)}$ flow travels from u to v if $u \to v$



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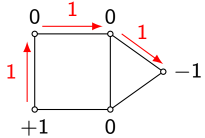
$$\left(div \cdot f\right)_v = \sum_{u:u \to v} f_{(u,v)} - \sum_{w:v \to w} f_{(w,v)}$$

- Say that flow f routes demand d if $div \cdot f = d$



Linear routing schemes

- A linear routing scheme is a function R such that
  - R maps a demand to a flow that routes it, and
  - $R : \mathbb{R}^V \to \mathbb{R}^E$ is linear

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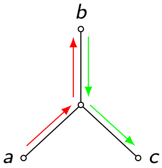
- Examples
  - Routing along a spanning tree, or
  - Electric routing



Sets of demands and flows

- Write a set of demands $\{d_i \in \mathbb{R}^V\}_{i=1,\dots,k}$ as $\oplus_i d_i \in \mathbb{R}^{V \times k}$
- Similarly, a set of flows $\{f_i \in \mathbb{R}^E\}_{i=1,\dots,k}$ as $\oplus_i f_i \in \mathbb{R}^{E \times k}$

* $x \oplus y$ means concatenate column vectors x and y into $\begin{pmatrix} | & | \\ x & y \\ | & | \end{pmatrix}$.

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- Say flows $\oplus_i f_i$ route demands $\oplus_i d_i$ if $div \cdot (\oplus_i f_i) = \oplus_i d_i$
- Simply means:
  - Flow f_i routes demand d_i for all i, by applying "flow f routes demand d if $div \cdot f = d$"

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Congestion and norms

- For a set of flows $F = \oplus_i f_i$ the congestion equals
  - The traffic on the most loaded edge, or
  - $\|F^*\|_{1\to 1}$ where $\|A\|_{1\to 1} := \sup_x \dfrac{\|Ax\|_1}{\|x\|_1}$
- Notably, congestion is a norm (over $\mathbb{R}^{E\times\infty}$)
- Abbreviate it as $\|F\|$

Worst-case demands

- Recall, for a scheme R, ratio is $\eta_R := \max_D \frac{\|R(D)\|}{\|opt(D)\|}$
- W.L.O.G. $\|opt(D)\| = 1$

Theorem For all such D, $\|R(D)\| \leqslant \|R(D_{\text{worst}})\|$, where D_{worst} demands one unit of flow between endpoints of every edge in G.

- So, $\eta_R = \|R(D_{\text{worst}})\|$

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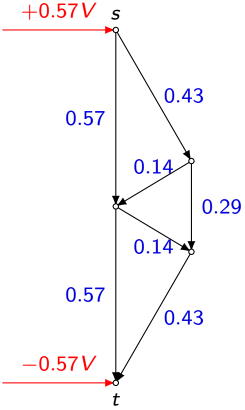
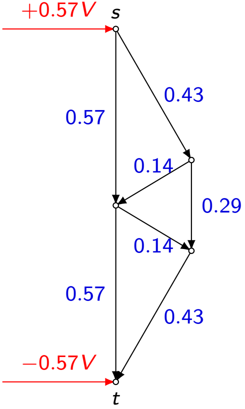
Electric flow



- How to map demand to electric flow?

$$\nabla \cdot L^{\dagger} : \mathbb{R}^{V} \to \mathbb{R}^{E}$$

Electric flow



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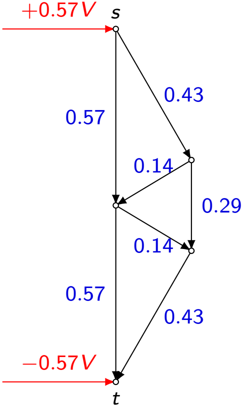
$$\nabla \cdot L^{\dagger} : \mathbb{R}^{V} \rightarrow \mathbb{R}^{E}$$

- where $\nabla : \mathbb{R}^{V} \rightarrow \mathbb{R}^{E}$ maps vertex potentials to edge potential differences

Electric flow



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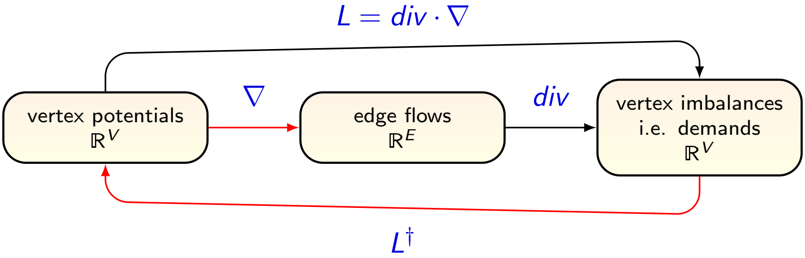
$$\nabla \cdot L^{\dagger} : \mathbb{R}^V \rightarrow \mathbb{R}^E$$

- where $\nabla : \mathbb{R}^V \rightarrow \mathbb{R}^E$ maps vertex potentials to edge potential differences
- and $L := div \cdot \nabla$

Derivation of electric flow



- Ohm's law: edge flow = potential difference * edge conductance

- ∇ maps vertex potentials to edge flow (Ohm's law)
- div maps edge flows to vertex flow imbalance (a.k.a. demand)

- $\nabla \cdot L^\dagger$ maps demand to edge flows

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Ratio of electric routing

- Electric routing operator is $\nabla \cdot L^{\dagger}$
- Worst-case demands are D_{worst} (by theorem)

  One unit of demand between the endpoints of every edge in G.
- So, competitive ratio equals $\|\nabla \cdot L^{\dagger} \cdot D_{\text{worst}}\|$

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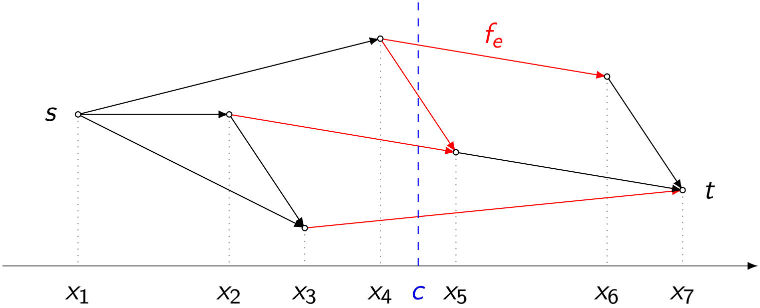
  $= \max_{s \neq t} \|\nabla \cdot L^{\dagger}(\mathbb{1}_s - \mathbb{1}_t)\|_1$

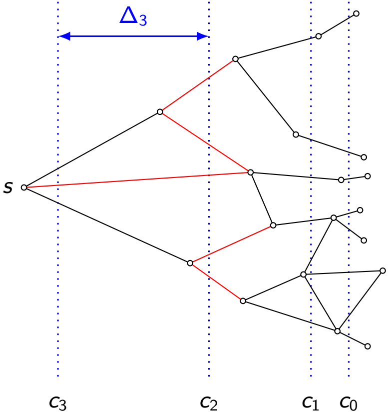
Laplacian $\ell_1 \to \ell_1$ norm bound

Theorem:
$\left\|\nabla L^{\dagger}(\mathbb{1}_s - \mathbb{1}_t)\right\|_1 \leqslant O(\log n)$, if $\min_{S \subseteq V} \frac{|E(S,S^{\complement})|}{\min|S|,|S^{\complement}|} = O(1)$.

- Think $(\mathbb{1}_s - \mathbb{1}_t) \overset{L^{\dagger}}{\longmapsto} x \overset{\nabla}{\longmapsto} f$, and ask $\|f\|_1 \leqslant ?$
- Local property: Sum of edge lengths on any cut equals 1



Rounding argument



k_i = number of edges cut by c_i

n_i = number of vertices to left of c_i

Idea Make a few cuts, then upper-bound total edge length by (scaled) edge length on cuts.

- **Invariant** $k_i = \Theta(n_i)$
- **Cut spacing** $\Delta_{i+1} = |c_i - c_{i+1}| =$ twice the avg. edge length on c_i

$\Rightarrow$ $k_{i+1} \leqslant \theta k_i$ where $0 < \theta < 1$ const.

$\Rightarrow$ $\Delta_{i+1} \geqslant \dfrac{\Delta_i}{\theta}$

- Using $\sum_i \Delta_i \leqslant \lambda^{-1} = O(1)$
- Conclude at most $O(\log n)$ cuts

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Remarks

- Computation
  - Approximate $L^{\dagger}$ by low-degree power-series polynomial in L
  - Multiplication by L is one distributed step
- Potential perturbation
  - Computed potentials are not exact
  - Theorem Electric flow under perturbed potentials as good
- Laplacian symmetrization to get degree independence

Thank you!

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